The University of Texas at Austin
Dept. of Electrical and Computer Engineering
Midterm #2

Date: November 18, 2010

Course: EE 313 Evans

Name: Set, Solution
Last, First

- The exam is scheduled to last 75 minutes.
- Open books and open notes. You may refer to your homework assignments and homework solution sets.
- **Power off all cell phones**
- You may use any standalone calculator or other computing system, i.e. one that is not connected to a network.
- All work should be performed on the quiz itself. If more space is needed, then use the backs of the pages.
- **Fully justify your answers unless instructed otherwise.**

| Problem | Point Value | Your score | Topic |
|---|---|---|---|
| 1 | 24 | | Differential Equation |
| 2 | 21 | | Integrator |
| 3 | 24 | | Transfer Functions |
| 4 | 21 | | Quadrature Amplitude Modulation |
| 5 | 10 | | Fourier Series |
| Total | 100 | | |

**Problem 2.1** Differential Equation. *24 points.*

For a continuous-time linear time-invariant (LTI) system with input $x(t)$ and output $y(t)$ is governed by the differential equation

$$\frac{d^2}{dt^2}y(t)+5\frac{d}{dt}y(t)+6y(t)=x(t)$$

for $t \geq 0^-$.

(a) Find the transfer function in the Laplace domain. *6 points.*

Because the system is LTI, $y(0^-)=0$ and $y'(0^-)=0$.

$$s^2\,Y(s) + 5s\,Y(s) + 6\,Y(s) = X(s)$$

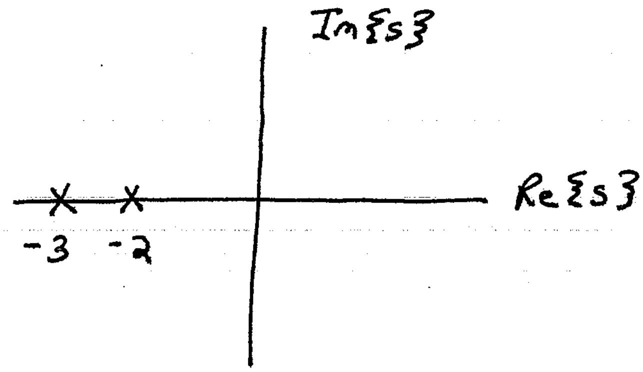
$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)} \qquad \mathrm{Re}\{s\} > -2$$

(b) Draw the pole-zero diagram in the Laplace domain. What are the pole location(s)? What are the zero location(s)? *6 points.*

There are no zeros.

Poles are located at

$s=-2$ and $s=-3$




(c) Find the impulse response. *6 points.*

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2} + \frac{-1}{s+3}\right\}$$

$$h(t) = e^{-2t}u(t) - e^{-3t}u(t)$$

(d) Give a formula for the step response of the system in the time domain. *6 points.*

Let $x(t) = u(t)$. $X(s) = \frac{1}{s}$.

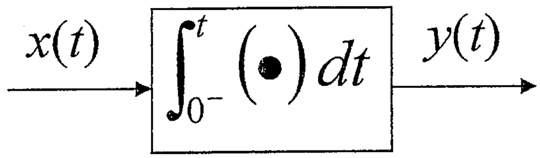
$$Y(s) = H(s)X(s) = \frac{1}{s(s+2)(s+3)}$$

$$= \frac{\frac{1}{6}}{s} + \frac{-\frac{1}{2}}{s+2} + \frac{\frac{1}{3}}{s+3}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{1}{6}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{3}e^{-3t}u(t)$$

**Problem 2.2** Integrator. *21 points.*

A continuous-time linear time-invariant (LTI) integrator is shown on the right. The initial condition $y(0^-) = 0$ for LTI.




(a) For the integrator above, give formulas for the impulse response $g(t)$, the transfer function in the Laplace domain $G(s)$, and the frequency response $G_{freq}(\omega)$ or $G_{freq}(f)$. *9 points.*

Impulse response: Let $x(t) = \delta(t)$. $y(t) = \int_{0^-}^{t} \delta(t)\, dt = u(t)$

Transfer function: $G(s) = \mathcal{L}\{g(t)\} = \frac{1}{s}$ $Re\{s\} > 0$

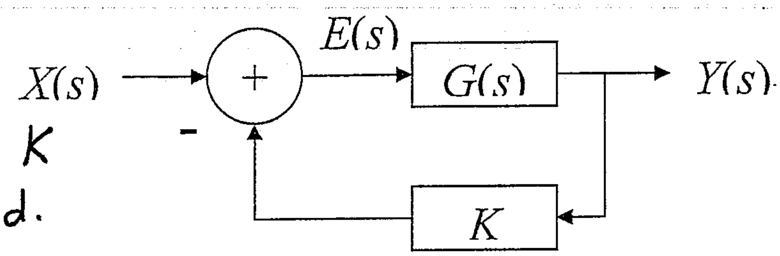
Frequency response: $G(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$

(b) Is the integrator bounded-input bounded-output (BIBO) stable? Why or why not? *3 points.*

No. Reason #1: Pole has to be in left-hand plane in Laplace domain for BIBO stability. Pole is at $s = 0$.

Reason #2: $\int_{-\infty}^{\infty} |g(t)|\, dt < \infty$ for BIBO stability. $\int_{-\infty}^{\infty} |u(t)|\, dt = \int_{0}^{\infty} dt$ which is unbounded.

(c) Consider the following LTI feedback system using the integrator building block, where $G(s)$ represents the LTI integrator and $K$ represents a scalar gain under computer control.




Assume that K is real-valued.

What is the transfer function $H(s)$? *3 points.*

$$H(s) = \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + K\, G(s)} = \frac{\frac{1}{s}}{1 + K\frac{1}{s}} = \frac{1}{s + K}$$

For what values of $K$ is the system BIBO stable? *3 points.*

Pole is at $s = -K$. System is BIBO stable if pole is in left-hand side of Laplace domain: $K > 0$

When system is BIBO stable, what kind of frequency selectivity does the system have? Lowpass, highpass, bandpass, bandstop, notch or all-pass? *3 points.*

$H_{freq}(\omega) = \frac{1}{j\omega + K}$ and $K > 0$ for BIBO stability.

$H_{freq}(0) = \frac{1}{K}$ $\lim_{\omega \to \infty} |H_{freq}(\omega)| = 0$. Pole location on negative real axis of Laplace domain means a lowpass filter.

**Problem 2.3** Transfer Functions. *24 points.*

A causal linear time-invariant (LTI) continuous-time system has the following transfer function in the Laplace transform domain:

$$H(s) = \frac{s-1}{s+1}$$

(a) Find the corresponding differential equation using $x(t)$ to denote the input signal and $y(t)$ to denote the output signal. Give the minimum number of initial conditions, and their values. *6 points.*

$$\frac{Y(s)}{X(s)} = \frac{s-1}{s+1} \Rightarrow \begin{aligned} (s+1)\,Y(s) &= (s-1)\,X(s) \\ s\,Y(s) + Y(s) &= s\,X(s) - X(s) \\ y'(t) + y(t) &= x'(t) - x(t) \end{aligned}$$

Initial conditions and values for LTI: $y(0^-) = 0$ and $x(0^-) = 0$

(b) Is the system bounded-input bounded-output (BIBO) stable? Why or why not? *6 points.*

For BIBO stability, all poles must be in the left-hand plane in the Laplace domain. The lone pole is at $s = -1$. System is BIBO stable.

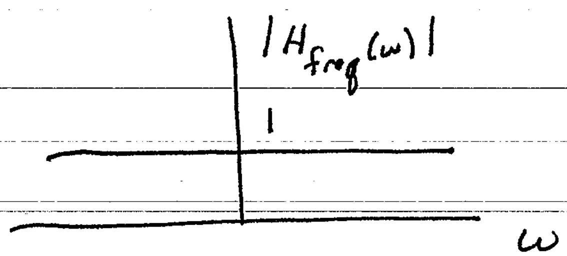
(c) Give a formula for the frequency response. *6 points.*

Because the system is BIBO stable,

$$H_{freq}(\omega) = H(s)\Big|_{s=j\omega} = \frac{j\omega - 1}{j\omega + 1}$$

(d) Plot the magnitude of the frequency response and describe the system's frequency selectivity (lowpass, highpass, bandpass, bandstop, notch or all-pass). *6 points.*

$$|H_{freq}(\omega)| = \frac{|j\omega - 1|}{|j\omega + 1|} = \frac{\sqrt{1+\omega^2}}{\sqrt{1+\omega^2}} = 1$$




All-pass.

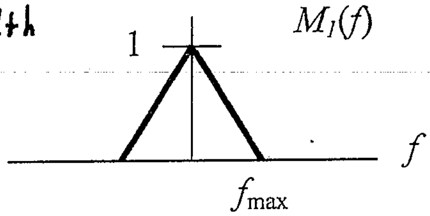
Hint: Use solid lines for real-valued amplitudes, and dashed lines for imaginary-valued amplitudes.

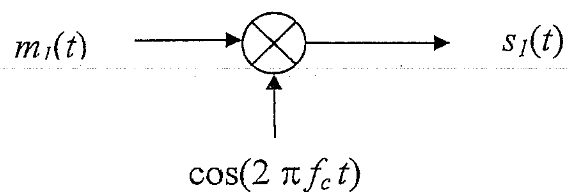
**Problem 2.4** Quadrature Amplitude Modulation. *21 points.*

Quadrature amplitude modulation uses cosine modulation and sine modulation together to use bandwidth more efficiently than using cosine modulation alone. Assume $f_c > f_{\max}$.

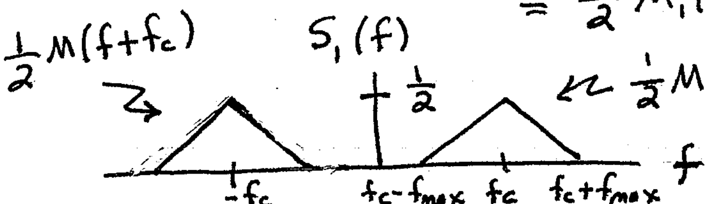
(a) For amplitude modulation using the cosine below, draw the spectrum $S_1(f)$. What is the transmission bandwidth? *6 points.*

Bandwidth of $M_1(f)$ is $f_{max}$



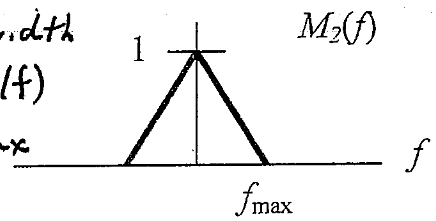

$$\mathcal{F}\{m_1(t)\cos(2\pi f_c t)\} = M_1(f) * \left(\tfrac{1}{2}\delta(f+f_c) + \tfrac{1}{2}\delta(f-f_c)\right)$$
$$= \tfrac{1}{2}M_1(f+f_c) + \tfrac{1}{2}M(f-f_c)$$

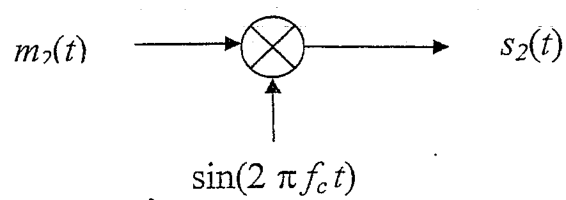



Transmission Bandwidth is $2f_{max}$

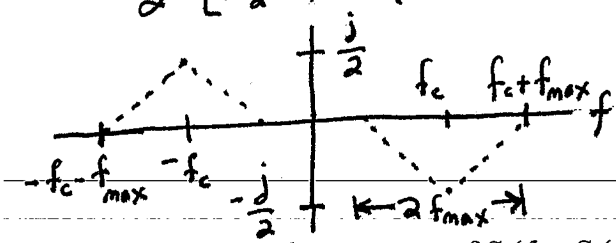
(b) For amplitude modulation using the sine below, draw the spectrum $S_2(f)$. *6 points.*

Bandwidth of $M_2(f)$ is $f_{max}$



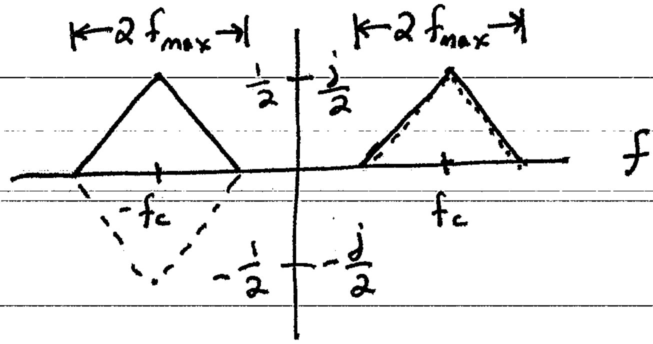




$$\mathcal{F}\{m_2(t)\sin(2\pi f_c t)\} = M_2(f) * \left(\tfrac{j}{2}\delta(f+f_c) - \tfrac{j}{2}\delta(f-f_c)\right)$$
$$= \tfrac{j}{2}M(f+f_c) - \tfrac{j}{2}M(f-f_c) = S_2(f)$$




Transmission Bandwidth is $2f_{max}$.

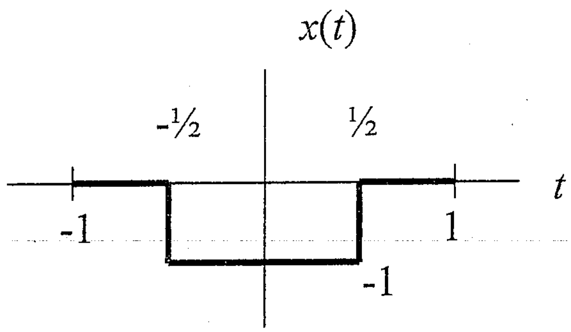
(c) Draw the spectrum of $S_1(f) - S_2(f)$. How would this more efficiently use transmission bandwidth than using amplitude modulation by a cosine? *9 points.*




Transmission Bandwidth is $2f_{max}$. Quadrature amplitude modulation obtained by $s_1(t) - s_2(t)$ transmits, in the same bandwidth, twice the number of messages.

**Problem 2.5** Fourier Series. *10 points.*

Compute the Fourier series according to its definition of the following signal:




$x(t) = -\text{rect}(t)$

The fundamental period $T_0$ is 2 s. $f_0 = \frac{1}{2}$ Hz and $\omega_0 = \pi$ rad/s.

$x(t)$ is 0 half of the period, and -1 the other half : $a_0 = \frac{1}{2}$

$x(t)$ is even symmetric ! $b_n = 0$

$$a_n = \frac{2}{T_0} \int_{-\frac{1}{2}T_0}^{\frac{1}{2}T_0} x(t) \cos(n\omega_0 t)\, dt = \int_{-1}^{1} -\text{rect}(t) \cos(n\omega_0 t)\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} (-1) \cos(n\omega_0 t)\, dt = -\frac{1}{n\omega_0} \sin(n\omega_0 t) \Big|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= -\frac{1}{n\omega_0} \left( \sin\left(\frac{\pi}{2} n\right) - \sin\left(-\frac{\pi}{2} n\right) \right)$$

$$= -\frac{2}{n\pi} \sin\left(\frac{\pi}{2} n\right)$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi t)$$